Systoles and Parameterization of Genus 2 Surfaces

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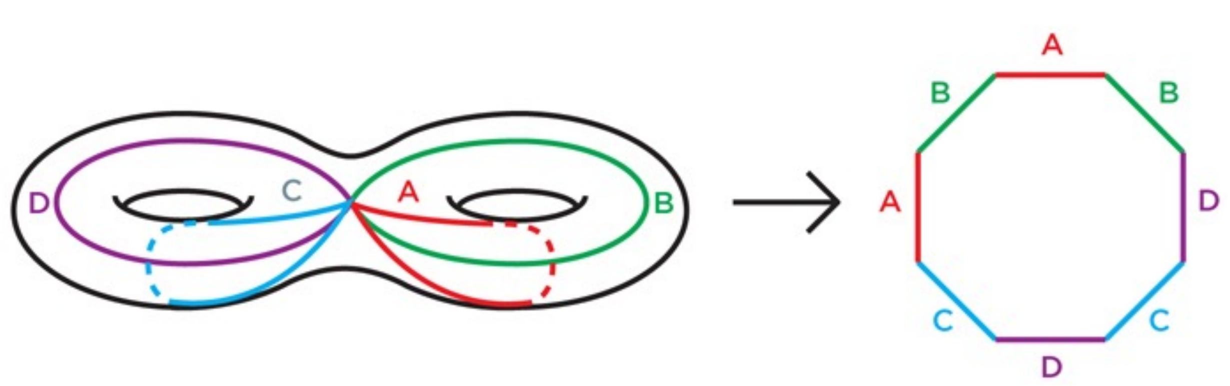
Introduction

Hyperbolic geometry, discovered in the 19th century, is a non-Euclidean geometry with constant negative curvature. Hyperbolic geometry can be used as a tool to study genus 2 surfaces because it is the natural geometry on surfaces of genus greater than or equal to 2 ($g \geq 2$). All hyperbolic genus two surfaces can be represented as octagons in the Poincare Disc model, and certain octagons in the hyperbolic disc can be glued to produce hyperbolic genus two surfaces. The standard gluing pattern of the octagons in the hyperbolic disc is 12123434 (gluing alternating sides together), as illustrated in the image below. There are also other gluing pattern that can be used. By the process of cut and paste, all these representations of genus 2 surfaces can be transformed between two different gluing patterns.

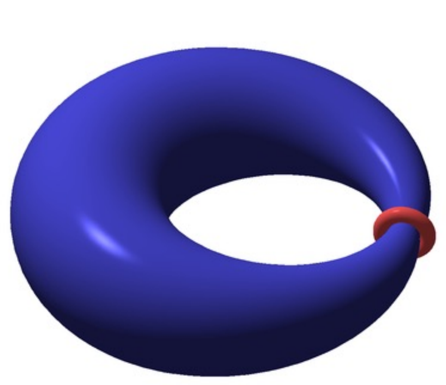
The goal is to find a hyperbolic genus 2 surface that has maximal systole.

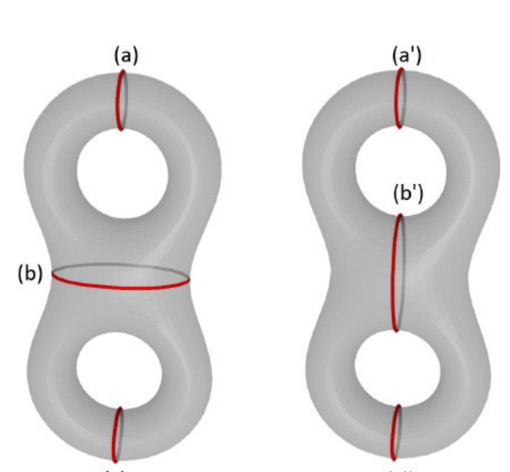
This kind of geometric optimization problem has potential applications in many fields. One example of an application is to design a valve of maximal efficiency in materials and biomedical engineering.



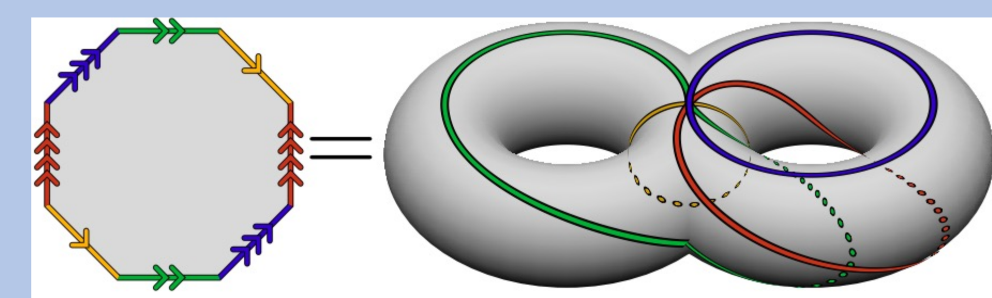
Systole

Given a closed surface, its systole is defined to be the least length of a loop that cannot be contracted to a point on the surface. The systole of a hyperbolic surface is the length of any of its shortest closed geodesics.





The Bolza Surface



Theorem: (Jenni, 1981)
The Bolza surface is the only surface that *globally* maximizes the systole among all genus 2 hyperbolic surfaces.

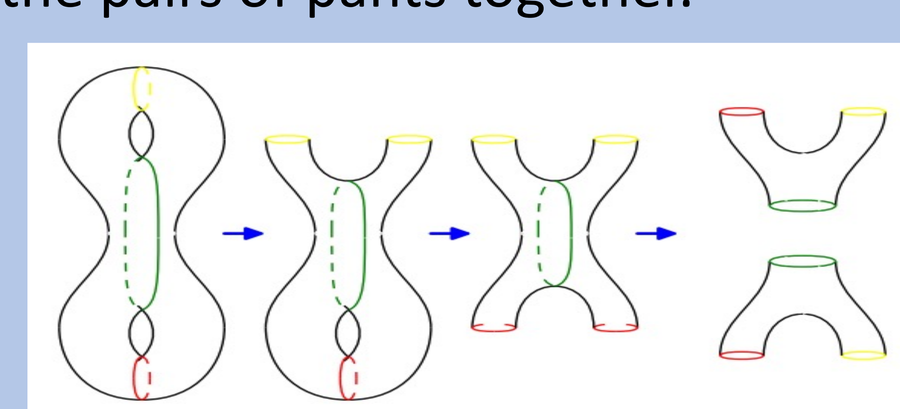
The theorem was first proved in 1981 by F.W. Jenni. Later on, P. Schmutz proved that the Bolza surface is the unique hyperbolic genus 2 surface that *locally* maximizes the systole in 1993.

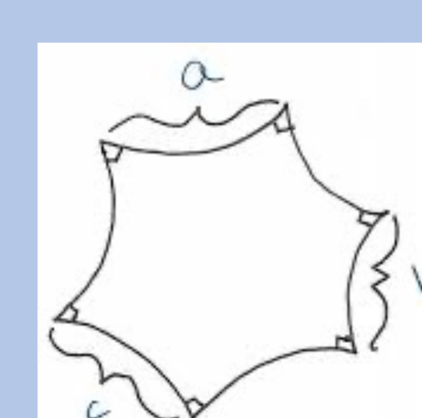
The Bolza surface follows the gluing pattern of 12341234, which glues each side to its opposite side to form a genus 2 surface. The octagon of the Bolza surface has 8 equal sides with 8 equal interior angles, with each angle 45°.

Fenchel-Nielsen coordinates

Fenchel-Nielsen Coordinates are coordinates that describe genus 2 surfaces with 6 parameters, written as (l_1, θ_1, l_2, θ_2, l_3, θ_3).

A genus 2 surface can be divided into two congruent pairs of pants by cutting along the middle. Each pair of pants can be further divided into two congruent right angled hexagons, so there are a total of 4 congruent right angled hexagons. There exists a unique right angled hexagon with edges $\frac{l_1}{2}, \frac{l_2}{2}, \frac{l_3}{2}$ as the three non-adjacent sides. The three angles θ_1, θ_2, θ_3 are the degrees of rotation when gluing the pairs of pants together.





Conclusion

All genus 2 surfaces can be represented as octagons in the hyperbolic plane, and can be represented by different gluing patterns.

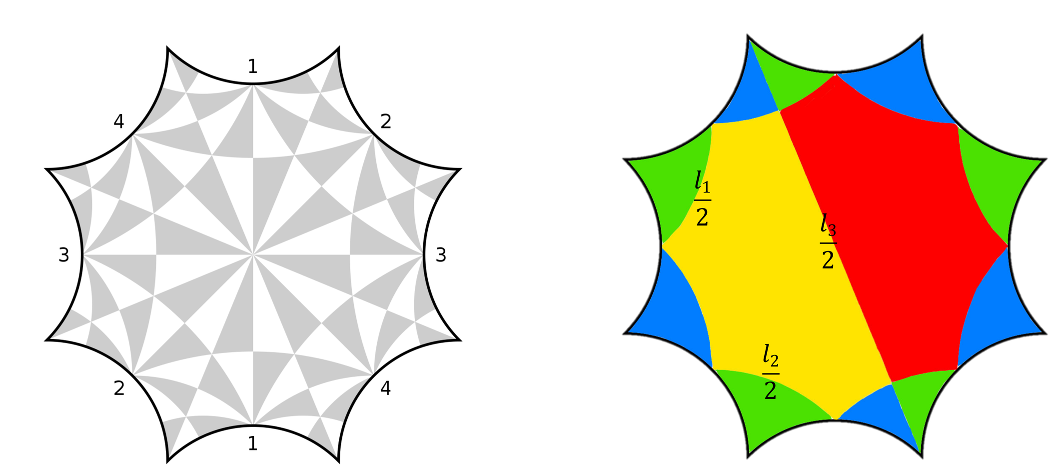
The Bolza surface is the genus 2 surface that maximizes the systole. The Fenchel-Nielsen parameters of the Bolza Surface are (l_1, 0, l_1, 0, l_2, 1/2), where we have:
$l_1 = 2 \operatorname{arcosh}(1+\sqrt{2}) \approx 3.05714$
$l_2 = 2 \operatorname{arcosh}(3+2\sqrt{2}) \approx 4.8969$

Notice that l_1 is shorter than l_2, so l_1 is the systole of the surface.

In the image below, there are 4 regions of different colors. These regions give 4 congruent right angles hexagons.

The decomposition of the Bolza surface into hexagons required to get the Fenchel-Nielsen coordinates, and the Fenchel-Nielsen parametrizations could be used to prove the theorem that the Bolza surface locally maximizes the systole.

By varying parameters of the Fenchel-Nielsen coordinates, the systole gets smaller. For example, decreasing the 1st or 3rd parameter of the coordinates clearly leads to a smaller systole. In the future we aim to prove that all variations of the Fenchel-Nielsen coordinates lead to a smaller systole.



Reference:
[1] F.W. Jenni, Über das Specktrum des Laplace-Operators auf einer Schar kompakter Riemannscher Flächen, Dissertation, Basel 1981.
[2] P. Schmutz, Riemann surfaces with shortest geodesic of maximal length, Geom. Funct. Anal. 3 (1993), no. 6, 564-631.