The Brunel Operator

& Pointwise Convergence of Cesàro Averages

I. Assani, S. McMahon, S. Schmidt, R.S. Hallyburton, C. Schoone

THE UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Problem Setup

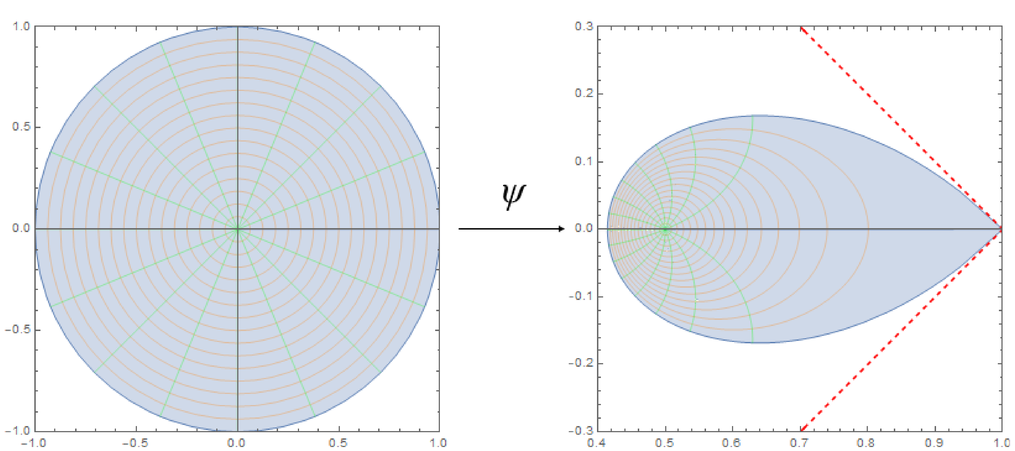
- A linear operator $T: E \to E$ on a Banach space E is *power-bounded* (resp. *Cesàro bounded*) if $\|T^n\| \le C$ (resp. $\|T + T^2 + \cdots + T^n\| \le nC$) for all positive integers *n* and for some constant *C*.
- For a power-bounded operator T the *Cesàro averages* $M_n(T)x$ converge in norm to x_* with $Tx_* = x_*$:

$$\lim_{n\to\infty} \left\| x_* - \frac{1}{n}\sum_{k=1}^{n} T^k x \right\| = 0.$$

- When $E = L^p(X,\mu)$ is an L^p space (a space of functions whose p^{th} power is integrable), a major question in ergodic theory is when the averages also converge pointwise to the same limit for "almost every" $x \in X$:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} T^k f(x) = f_*(x) \text{ for } \mu - \text{almost every } x \in X.$$

- Currently open question: if $T: L^p(X,\mu) \to L^p(X,\mu)$ is Cesàro bounded and *positive* ($f \ge 0 \Rightarrow Tf \ge 0$), do the Cesàro averages of T converge pointwise almost everywhere?



The Brunel Operator

- For each positive integer *n*, expand the function ψ^n as a power series at the origin:

$$\psi^n(z) = \left(\frac{1-\sqrt{1-z}}{z}\right)^n = \sum_{p=0}^{\infty} \alpha_p^{(n)} z^p.$$

- For a Cesàro bounded operator *T*, the *Brunel operator* associated to *T* is:

$$A(T) = \sum_{p=0}^{\infty} \alpha_p^{(1)} T^p.$$

- In our paper "New Estimates on the Brunel Operator", we prove that when *T* is Cesàro bounded, the Brunel operator is power-bounded and satisfies a decay condition known as the *Ritt condition*:

$$n\|A^{n+1}(T) - A^n(T)\| \le \left(33 + 4\sqrt{3}\right) \sup_{n\in\mathbb{N}} \|M_n(T)\|.$$

- Unlike previous work on the Brunel operator (due to Brunel, Emilion, Lootgieter), proofs are facilitated primarily by estimates on the series coefficients. Main estimates:

$$\sup_{n\in\mathbb{N}} n \sum_{p=0}^{\infty} \left|\alpha_p^{(n+1)} - \alpha_p^{(n)}\right| \le \frac{33}{2}, \sup_{n\in\mathbb{N}} \sum_{p=0}^{\infty} \left|\alpha_p^{(n)} - \alpha_{p+1}^{(n)}\right| (p+1) \le 3 + \frac{5}{\sqrt{\pi}}.$$

Main Results and Future Directions

- Cesàro averages of *T* converge (in norm or pointwise) to x_* if and only if *powers* of the Brunel operator converge (in norm or pointwise) to x_*
- Problem of pointwise convergence for positive Cesàro bounded operators reduced (via the Brunel operator) to pointwise convergence for positive power-bounded operators satisfying Ritt condition
- Pointwise convergence of the powers of the Brunel operator is still open – current work seeks to sharpen estimates on the coefficients $\alpha_p^{(n)}$ in order to assess pointwise convergence