The Brunel Operator
& Pointwise Convergence of Cesàro Averages

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Problem Setup

• A linear operator $T: E \to E$ on a Banach space $E$ is power-bounded (resp. Cesàro bounded) if $\|T^n\| \leq C$ (resp. $\|T + T^2 + \cdots + T^n\| \leq nC$) for all positive integers $n$ and for some constant $C$.

• For a power-bounded operator $T$ the Cesàro averages $M_n(T)x$ converge in norm to $x_*$ with $T_{x_*} = x_*:
\lim_{n \to \infty} \left\| x_* - \frac{1}{n} \sum_{k=1}^n T^k x \right\| = 0.$

• When $E = L^P(X, \mu)$ is an $L^P$ space (a space of functions whose $p^{th}$ power is integrable), a major question in ergodic theory is when the averages also converge pointwise to the same limit for “almost every” $x \in X:
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n T^k f(x) = f_*(x)$ for $\mu$ – almost every $x \in X$.

• Currently open question: if $T: L^P(X, \mu) \to L^P(X, \mu)$ is Cesàro bounded and positive ($f \geq 0 \Rightarrow Tf \geq 0$), do the Cesàro averages of $T$ converge pointwise almost everywhere?

The Brunel Operator

• For each positive integer $n$, expand the function $\psi^n$ as a power series at the origin:
$$\psi^n(z) = \left(1 - \frac{1 - z}{z}\right)^n = \sum_{p=0}^{\infty} \alpha_p^{(n)} z^p.$$

• For a Cesàro bounded operator $T$, the Brunel operator associated to $T$ is:
$$A(T) = \sum_{p=0}^{\infty} \alpha_p^{(1)} T^p.$$

• In our paper “New Estimates on the Brunel Operator”, we prove that when $T$ is Cesàro bounded, the Brunel operator is power-bounded and satisfies a decay condition known as the Ritt condition:
$$n\|A^{n+1}(T) - A^n(T)\| \leq \left(33 + 4\sqrt{3}\right) \sup_{n \in \mathbb{N}} \|M_n(T)\|.$$

• Unlike previous work on the Brunel operator (due to Brunel, Emilion, Lootgieter), proofs are facilitated primarily by estimates on the series coefficients. Main estimates:
$$\sup_{n \in \mathbb{N}} \left| \alpha_p^{(n+1)} - \alpha_p^{(n)} \right| \leq \frac{33}{2}, \sup_{n \in \mathbb{N}} \sum_{p=0}^{\infty} \left| \alpha_p^{(n)} - \alpha_p^{(n+1)} \right| (p + 1) \leq 3 + \frac{5}{\sqrt{\pi}}.$$

Main Results and Future Directions

• Cesàro averages of $T$ converge (in norm or pointwise) to $x_*$ if and only if powers of the Brunel operator converge (in norm or pointwise) to $x_*$. Problem of pointwise convergence for positive Cesàro bounded operators reduced (via the Brunel operator) to pointwise convergence for positive power-bounded operators satisfying Ritt condition.

• Pointwise convergence of the powers of the Brunel operator is still open – current work seeks to sharpen estimates on the coefficients $\alpha_p^{(n)}$ in order to assess pointwise convergence.