A Local Energy Estimate for Damped Wave Equations

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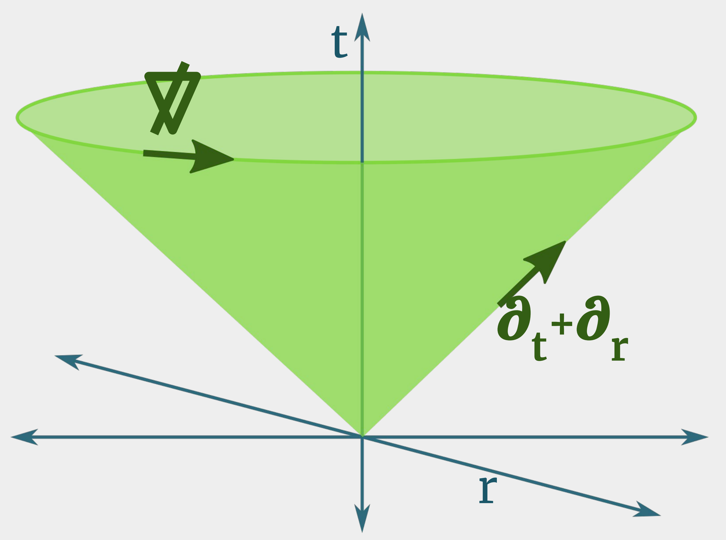
Background: Waves

Basic notations:

- Let u denote the wave function.
  - $u \in C^2(\mathbb{R}_+, \mathbb{R}^3)$
  - identically 0 outside the light cone
- Wave operator (d'Alembertian):
  $$\Box = \partial_t^2 - \Delta$$
- Decomposition of the gradient: $\nabla = \frac{x}{r}\partial_r + \not\nabla$
- Radial derivative: $\partial_r = \frac{x}{r} \cdot \nabla$
- Angular derivative: $\not\nabla$

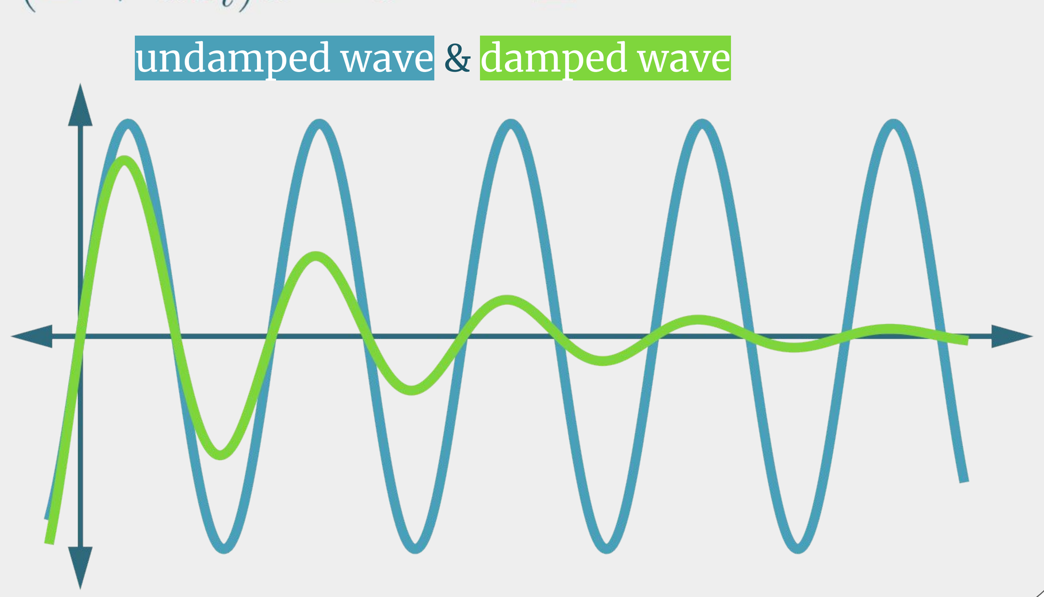
Waves that propagate at finite speeds may only occupy a region of spacetime called the "**light cone.**"



Derivatives **tangent** to the light cone are of particular interest for estimates.

Introducing Damping:

- Damping is a force that reduces the amplitude of oscillations (e.g. drag, electrical resistance)
- Equation of a free wave (no damping): $\Box u = 0$
- Equation of a linearly damped wave:
  $$(\Box + a\partial_t)u = 0 \qquad a \geq 0$$



Goal: Modify an existing r^p-weighted energy estimate [1][2][4] to account for damping.

Theorem. *Fix $0 < p < 1$ and let $a \geq 0$. Suppose $u \in C^2(\mathbb{R}_+ \times \mathbb{R}^3)$ and for every $t \in \mathbb{R}_+$, $r^{\frac{p}{2}}|u(t,x)| \to 0$ as $|x| \to \infty$. Then for $\Box + a\partial_t = 0$,*

$$\int_0^T \int_{\mathbb{R}^3} \langle r\rangle^{p-1} r^{-2} \left((\partial_t + \partial_r)(ru) \right)^2 dx\,dt + \int_0^T \int_{\mathbb{R}^3} \langle r\rangle^p r^{-1} \left|\not\nabla u\right|^2 dx\,dt$$
$$+ a \int_0^T \int_{\mathbb{R}^3} \langle r\rangle^p r^{-2} \left((\partial_t + \partial_r)(ru) \right)^2 dx\,dt + a \int_0^T \int_{\mathbb{R}^3} \langle r\rangle^p \left|\not\nabla u\right|^2 dx\,dt$$
$$+ a \int_0^T \int_{\mathbb{R}^3} \langle r\rangle^{p-2} u^2\, dx\,dt$$
$$\lesssim E[u](0)$$

Result: Discussion

Undamped:	$\int_0^T \int_{\mathbb{R}^3} \frac{\langle r\rangle^{p-1}}{r^2} \left((\partial_t + \partial_r)(ru) \right)^2 dx\,dt$	$\int_0^T \int_{\mathbb{R}^3} \frac{\langle r\rangle^p}{r} \left|\not\nabla u\right|^2 dx\,dt$	$\int_0^T \int_{\mathbb{R}^3} \frac{\langle r\rangle^{p-1}}{r^2} u^2\, dx\,dt$
Damped:	$\int_0^T \int_{\mathbb{R}^3} \frac{\langle r\rangle^p}{r^2} \left((\partial_t + \partial_r)(ru) \right)^2 dx\,dt$	$\int_0^T \int_{\mathbb{R}^3} \langle r\rangle^p \left|\not\nabla u\right|^2 dx\,dt$	$\int_0^T \int_{\mathbb{R}^3} \langle r\rangle^{p-2} u^2\, dx\,dt$

- Powers of *r* increase with damping, **improving** the estimate by increasing the lower bound.
  - This improvement is expected because damping provides an additional source of decay.
- Note that the range of the constant p does not show improvement. In previous literature [4] for the undamped case, the range of p was allowed to be $0 < p < 2$, while the damped case requires $0 < p < 1$.

Background: Energy

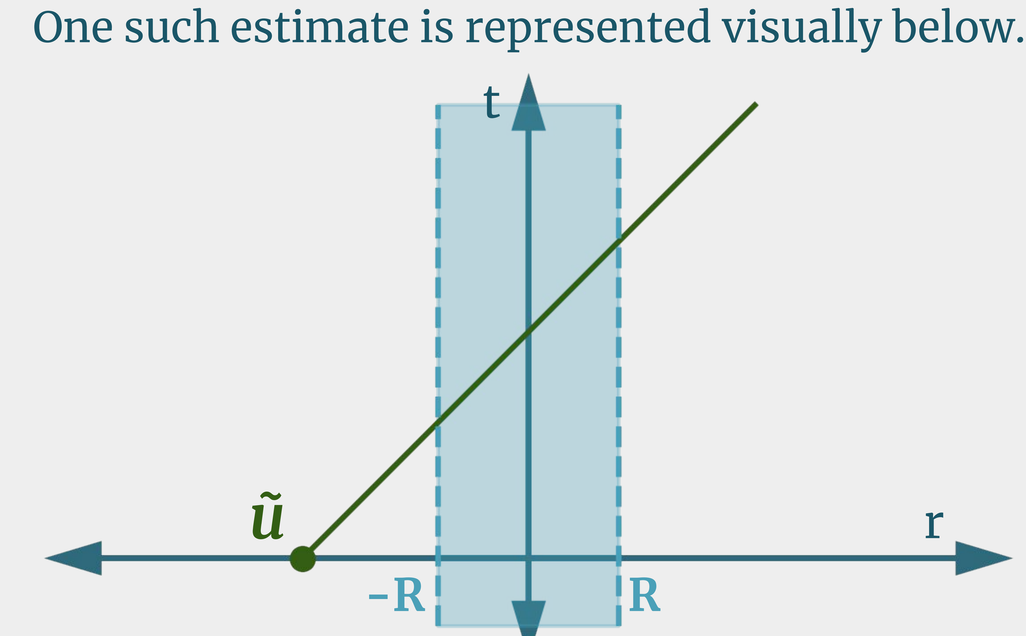
Energy is defined as

$$E[u](t) = \frac{1}{2}\int |u'(t,x)|^2\, dx$$

An **energy estimate** bounds derivatives of the wave function by initial energy.

Estimates:

- **Conservation of Energy** for Damped Waves:
  $$E[u](t) \lesssim E[u](0)$$
- **Morawetz Estimate** [5]:
  $$\int_0^T \int_{\mathbb{R}^3} \frac{|\not\nabla u|^2}{r}\, dx\,dt \lesssim E[u](0)$$
- **Weighted Estimate**:
  To improve the bound, an estimate can include some function as a weight [3,6].
  One such estimate is represented visually below.



For a packet $\tilde{u}$ of a solution, and radius *R* about the origin, $\tilde{u}$ spends no longer than *O(R)* in the radial region.

$$\int_0^T \int_{|x| \leq R} |\tilde{u}'(t,x)|^2\, dx\,dt \lesssim R\, E[u](0)$$
$$\Rightarrow \sup_R \frac{1}{R} \int_0^T \int_{|x| \leq R} |u'(t,x)|^2\, dx\,dt \lesssim E[u](0)$$

In our work, we modify an estimate [2,4] that uses an r^p weight [1], to be used for damped waves, as stated in the theorem.

References

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[6] Sterbenz, J.: Angular regularity and Strichartz estimates for the wave equation. Int. Math. Res. Not., (4):187–231, 2005. With an appendix by Igor Rodnianski.

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