An Exploration of Nonlinear Water Wave Generation

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Introduction to Water Waves

Traveling water wave problems have been of great interest to researchers for the past several centuries. Although accurate models have been developed for a variety of these problems, there still remain gaps in the theory where higher-order approximations can be used to generate models that better predict experimental results.

Current Models

Green-Naghdi Equations

$$\underline{u_t} + uu_x + \zeta_x = -\frac{1}{\eta}\Big[(\eta p_1)_x + p_1(b)b_x\Big]$$

$$\eta_t + (\eta u)_x = 0$$

Generalized-Boussinesq Equations

$$(\zeta_t - \alpha b_t) + \left[(1 - \alpha^2 b + \alpha\zeta)u\right]_x = 0$$

$$u_t + \alpha u u_x + \zeta_x = \frac{\epsilon^2}{3}u_{xxt} - \frac{\epsilon^2}{2}b_{xtt} - p_{0x}$$

Forced Korteweg-De Vries Equation

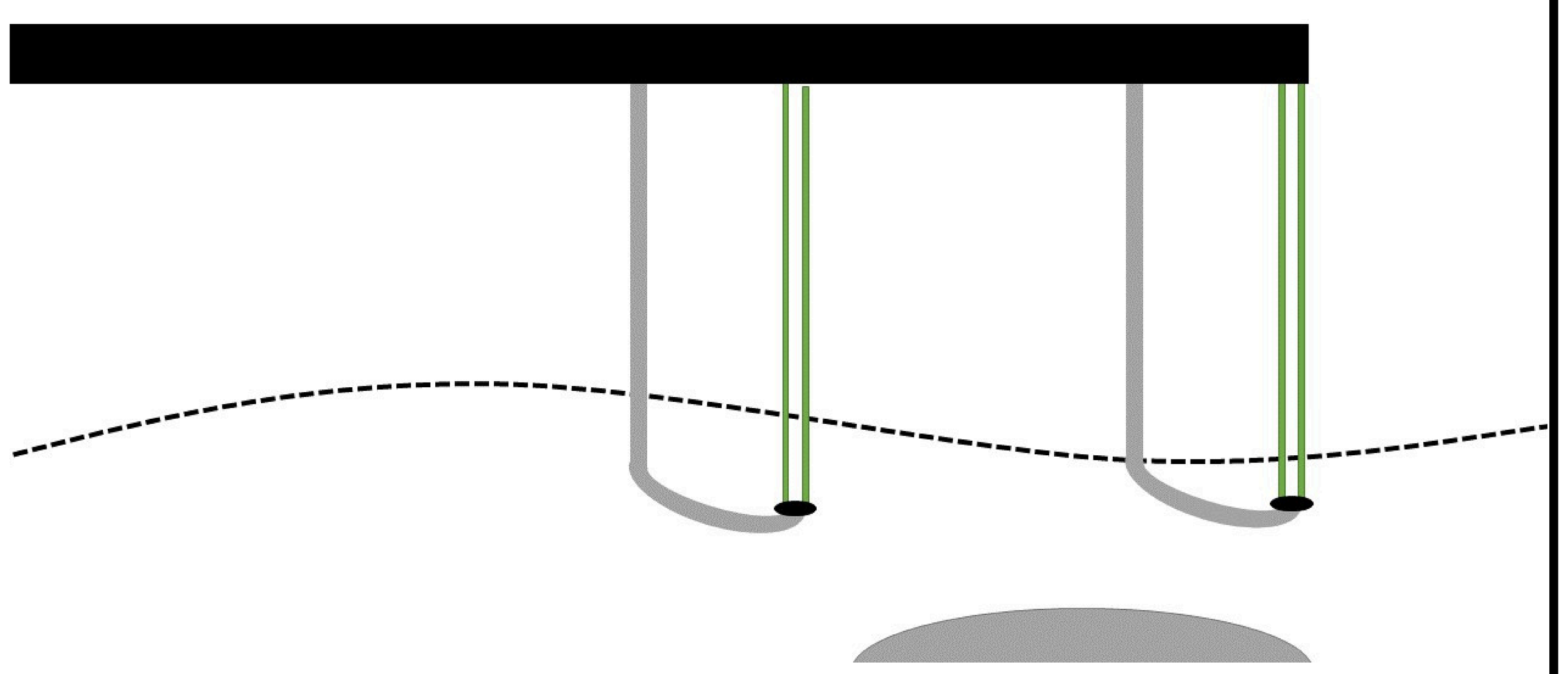
$$\zeta_t + (1 + \frac{3}{2}\zeta)\zeta_x + \frac{\epsilon^2}{6}\zeta_{xxx} = -\frac{1}{2}(p_{0x} + b_x)$$

The Green-Naghdi equations can be derived directly from the fundamental Euler Equations by perturbatively expanding the velocity and free surface displacement in the shallow water parameter and vertically averaging the results. The Generalized-Boussinesq equations then arise from an assumption of "weak nonlinearity," which allows for further expansion in the nonhydrostatic pressure terms of the Green-Naghdi equations. Finally, the unidirectional Forced Korteweg-De Vries equations result from yet another expansion, this time assuming a function relationship between fluid flow velcocity and free surface displacement. All of these are nonlinear partial differential equations, and can be used to predict shallow water flows over a topography.

Experimentation

Flow-over-topography experiments were conducted in the Joint Fluids Lab in a 36-meter-long wave tank. Waves were generated using a computer-controlled travelling bottom topography, which, under a change of reference frame, simulates constant velocity flow over a stationary topography. Water depth versus time data were collected using two stationary *Akamina AWP-24-3* capacitance wave gauges. This data allowed for visual and numerical analysis of the waves generated by the travelling topography, and the results of our experiments motivated our theoretcal work.

Depiction of Experimental Setup



Results

At near-critical flow speeds, clear upstream travelling waves begin to form, which can be accurately modelled by current wave equations which emit solitary wave solutions, such as fKdV. In supercritical experiments, we observed the formation of upstream hydraulic jumps. Although this is in agreement with previous experimental results, this type of wave profile cannot be predicted by the current models of shallow water flow and demonstrates the need for expansion into higher-order models.

Theory

Taking cues from the derivation of fKdV, we assumed a functional relation between flow velocity and free surface displacement, and expanded perturbatively in terms of small parameters representing smallness of wave amplitude and shallowness of water depth.

$$\zeta = u + \frac{1}{4}\alpha u^2 - \frac{1}{6}\epsilon^2 u_{xx} + \frac{1}{2}\alpha b + \alpha^2 L(u) + \epsilon^2 \alpha M(u)$$

In order to determine the final two unknown functions, we inserted this expansion back into the Green-Naghdi equations and used the extra freedom to equate the two equations up to $O(\alpha\epsilon^2)$. This generated our final model, which is presented below.

$$u_t + u_x + \frac{3}{2}\alpha u u_x - \frac{1}{6}\epsilon^2 u_{xxt} + \frac{1}{2}\alpha b_x + \alpha^2\Big(\frac{1}{4}b_x(\delta - u) - \frac{1}{4}(bu)_x\Big) =$$

$$\alpha\epsilon^2\Big(-\frac{23}{24}u_x u_{xx} - \frac{1}{6}u u_{xxx} - \frac{3}{4}b_{xxx}\Big)$$

This model incorporates several extra nonlinear terms that do not appear in fKdV, which is a promising sign that this model could emit solutions which can be used to analyze hydraulic jumps. Although shallow-water models technically should not generate these solutions, heuristic analysis of the right-side terms shows the possibility of reverse diffusive behavior. This, in combination with the forcing and nonlinear terms littered throughout the equation, could create the specific type of breakdown needed for solutions with jumps.

Several adjustments would still be needed to make this equation a usable model of shallow flow over a topography, but nevertheless, it contains many interesting properties and opens new avenues for analytical and numerical study of this problem.

Future Work

Numerical simulations are currently being developed to determine if our new model is better at predicting shallow flows over a topography than previous models. Additionally, we would like to establish a Hamiltonian structure for our model and adjust the coefficients to follow from a variational principle. Finally, more experiments will be performed using fine adjustments to identify the exact placement of critical points in flow behavior.

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